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ABSTRACT

The instabilities of some spatially and/or time-periodic flows are discussed, in particular, flows with curved streamlines which can support Taylor-Görtler vortices are described in detail. The simplest flow where this type of instability can occur is that due to the torsional oscillations of an infinitely long circular cylinder. For more complicated spatially varying time-periodic flows, a similar type of instability can occur and is spatially localized near the most unstable positions. When nonlinear effects are considered it is found that the instability modifies the steady streaming boundary layer induced by the oscillatory motion. It is shown that a rapidly rotating cylinder in a uniform flow is susceptible to a related type of instability; the appropriate stability equations are shown to be identical to those which govern the instability of a Boussinesq fluid of Prandtl number unity heated time periodically from below.

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1. INTRODUCTION

Our main concern is with the nature of the centrifugal instability of time-periodic flows which interact with curved surfaces. It appears at this stage that theory and experiment here are in much closer agreement than is the case for the instability of flat Stokes layers. In the latter problem it seems that the instabilities which exist experimentally occur at such high Reynolds numbers that quasi-steady theory applies and the solution of the full time-dependent stability equations, being damped, are irrelevant. The reader is referred to the papers by Cowley and Kerczek, which appear in this volume, for a detailed discussion of the flat Stokes layer. In contrast to this situation, for curved Stokes layers the solutions of the time-dependent equation explain a great deal of the available experimental results, whilst the quasi-steady solutions of the equations are irrelevant except in the inviscid or high wavenumber viscous limits.

The existence of a Taylor-Görtler instability in the Stokes layers on a torsionally oscillating cylinder was demonstrated experimentally and theoretically by Seminara and Hall (1976). Since Rayleigh's criterion does not apply to time-dependent flows, there is no simple way of determining whether a given time-dependent basic flow is inviscidly unstable. Indeed, since Floquet theory can only be used for equations with time-periodic coefficients, unless the basic state is varying periodically in time, it is not even clear how to define instability.

The onset of instability predicted theoretically is consistent with the available experimental observations; however, at high Taylor numbers it is found experimentally that a secondary instability which progressively doubles the axial wavelengths occurs. At some stage in this process the flow acquires

an azimuthal dependence; there is as yet no adequate theoretical description of this regime, and it seems likely that only a fully numerical investigation will reveal its structure.

More recently the instability mechanism found by Seminara and Hall (1976) has been shown experimentally and theoretically to occur in a wide range of flows of practical importance. These flows vary (periodically) in time and at least two spatial directions and exhibit the so-called steady streaming phenomenon associated with the work of, for example, Rayleigh (1883), Schlichting (1932), Stuart (1966), and Riley (1967). In these flows the Reynolds stresses in the boundary layer of the first-order oscillatory viscous flow drive a mean motion which may or may not be confined to the Stokes boundary layer. In fact, the structure of the steady streaming depends on the size of R_s, a Reynolds number associated with the mean motion. The importance of R_s was first explained by Stuart (1966) who showed that for $R_s \gg 1$ the steady streaming decays to zero in an outer boundary layer of relative thickness $R_s^{+1/2}$. It turns out that the instability of the basic oscillatory flow also occurs for $R_s \gg 1$ and that the steady streaming is then driven by both the Reynolds stresses of the oscillatory flow and the instability. It is likely that in some situations the instability might cause the premature separation of the steady streaming boundary layer.

The thermal convection analogue of the torsionally oscillatory cylinder problem has apparently not yet been investigated. The instability equations for an infinite layer of fluid heated sinusoidally from above or below will be derived in this paper. It turns out that if the fluid has Prandtl number unity then the same equations govern the instability of the flow past a rapidly rotating cylinder in a uniform flow. The analogy between the problems

is similar to that which is known to exist between the steady rapidly rotating Taylor vortex and steady Bénard problems. Some preliminary results for the solution of the eigenvalue problem governing the instability of these flows are given.

The procedure adopted in the rest of this paper is as follows: In Section 2 the instability of the flow on a torsionally oscillating cylinder is discussed. In section 3 the generalization of this instability to spatially varying time-periodic flows is discussed. In Section 4 we discuss the analogy between the instability problems for the flow around a rapidly rotating cylinder and that driven by the time-periodic heating of a fluid layer. Finally, in Section 5 we draw some conclusions.

2. THE TORSIONALLY OSCILLATING CYLINDER PROBLEM

Consider the viscous flow induced by an infinitely long circular cylinder of radius R rotating about its axis with angular velocity $\Omega \cos \omega t$. The time-periodic boundary layer on the cylinder is taken to be small compared to R so that,

$$\left(\frac{\nu}{\omega}\right)^{1/2} << R,$$

and the velocity field $\underset{\sim}{u}_B$ is then given by,

$$\underset{\sim}{u}_B = \Omega R(0,\ V(\eta,\ \tau),\ 0),$$

where

$$\tau = \omega t, \quad \eta = \{r - R\}\left\{\frac{\nu}{2\omega}\right\}^{-1/2}$$

and

$$\bar{V}(\eta, \tau) = \cos\{\tau - \eta\}e^{-\eta}. \tag{2.1}$$

Thus $\bar{V}(\eta, \tau)$ is the usual Stokes velocity profile and it is convenient to define the Taylor-Görtler number T by

$$T = 2 \frac{\Omega^2 R}{\omega^{3/2} \nu^{1/2}}, \tag{2.2}$$

and it is assumed from now on that $T \sim 0(1)$.

The above time-periodic basic state is then perturbed to a disturbance which is periodic along the axis of the cylinder with wavelength $\frac{2\pi}{k}$ based on $(\frac{2\nu}{\omega})^{1/2}$ the boundary layer thickness. After some manipulation we find that the radial and azimuthal velocity components U, V satisfy the coupled system:

$$\mathcal{L}\{\frac{\partial^2}{\partial\eta^2} - k^2\}U = 2k^2\, T\bar{V}(\eta, \tau)V, \tag{2.2a}$$

$$\mathcal{L}V = \sqrt{2}\, \frac{\partial\bar{V}}{\partial\eta}\, U, \tag{2.2b}$$

where

$$\mathcal{L} \equiv \frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\,.$$

The system (2.2) must be solved subject to

$$U = V = \frac{\partial V}{\partial\eta} = 0, \quad \eta = 0, \tag{2.3}$$

and

$$U = V = 0, \quad \eta = \infty, \tag{2.4}$$

so that the no-slip condition is satisfied at the wall and the disturbance decays exponentially to zero at infinity. There is, of course, no justification for a quasi-steady approximation if T is $O(1)$ since the time dependence of the partial differential system is in no sense slow.

On the basis of Floquet theory we expect solutions of (2.2) of the form

$$U = e^{\sigma\tau}\, u(\eta, \tau), \quad V = e^{\sigma\tau}\, v(\eta, \tau)$$

where u, v are now periodic functions of τ. The Floquet exponent σ is a function of k and T and must be chosen such that the partial differential equations and boundary conditions are satisfied. If we write

$$u = \sum_{-\infty}^{\infty} e^{in\tau}\, u_n(\eta), \quad v = \sum_{-\infty}^{\infty} e^{in\tau}\, v_n(\eta),$$

then the eigenvalue σ is determined by the system

$$\left.\begin{aligned}
&\left[\frac{d^2}{d\eta^2} - k^2 - \sigma - 2in\right]\left[\frac{d^2}{d\eta^2} - k^2\right]u_n = \frac{k^2 T}{2}\left[e^{-\eta(1+i)}\, v_{n-1} + e^{-\eta(1-i)} v_{n+1}\right],\\
&\left[\frac{d^2}{d\eta^2} - k^2 - \sigma - 2in\right]v_n = \frac{-1}{\sqrt{2}}\left[(1+i)e^{-\eta(1+i)}\, u_{n-1} + (1-i)e^{-\eta(1-i)}\, u_{n+1}\right],\\
&u_n = v_n = \frac{du_n}{d\eta} = 0, \quad \eta = 0,\\
&u_n = v_n = 0, \quad \eta = \infty,
\end{aligned}\right\}\quad (2.5)$$

for $n = 0, \pm 1, \pm 2, \cdots$.

The eigenvalue problem (2.5) can be solved numerically by truncating n such that $|n| \leq N$ and finding $\sigma = \sigma(N, K, T)$. Numerical calculations by Seminara and Hall (1976) showed that instability occurs for

$$T > 164.42$$

and the critical value of the wavenumber is $k = 0.859$. The neutral curve calculated by Seminara and Hall is shown in Figure 1. The calculations also showed that $u_n = 0$ for n odd and that $v_n = 0$ for n even. These results are in excellent agreement with the experiments performed by Seminara and Hall (1976) and Barenghi, Park, and Donnelly (1980) who visualized the instability using dye visualization.

However, in both sets of experiments a secondary instability at a Taylor number about 30% above the critical was observed. The linear theory of Seminara and Hall showed that when instability occurs the function $v_0(\eta)$ is zero so that there is no mean flow around the cylinder. When the secondary instability occurs the vortices initially interact with their neighbours to generate larger vortices. Subsequently this larger set of vortices interact to produce an even larger set and this process appears to continue without any equilibrium state ever being reached. At some stage in this process the azimuthal velocity field develops a mean component and there are signs of nonaxisymmetry.

At this time very little is understood theoretically about how and why this secondary instability takes place. Seminara and Hall (1977) showed that the first mode bifurcates supercritically at the critical Taylor number. Hall (1981) showed, using an approximate approach, that an axisymmetric mode with

twice the wavelength of the first mode causes the first mode to lose stability very close to the experimentally observed critical Taylor number. However, it should be stressed that this approximate approach, which is similar to that recently used for mode interactions in spatially varying flows, cannot be rigorously justified; nevertheless, it would appear that it describes very well the first stages in the secondary instability regime. It is of interest to note that in the steady Taylor and Görtler vortex instabilities the onset of the secondary wavy vortex modes leads to a new equilibrium state.

The possible role of nonaxisymmetric modes in the secondary instability problem is still not understood; certainly it is known from the work of Duck and Hall (1981) that θ-dependent modes occur quite close to the axisymmetric critical Taylor number and are more stable than the axisymmetric one. Moreover, we can show that when nonaxisymmetric modes are present a steady streaming phenomenon occurs and the instability can no longer be confined to the boundary layer. In this case the velocity field decays to zero in an outer layer of relative thickness $R_s^{1/2}$ where R_s is a steady streaming Reynolds number.

It would appear that little further theoretical work on the secondary instability stage is possible until more precise experimental results become available. The flow visualization methods used previously are not sensitive enough to determine the detailed flow structure after the secondary instability occurs.

3. INSTABILITIES IN FLOWS EXHIBITING STEADY STREAMING

The prototype problem discussed in the previous section shows how a centrifugal instability mechanism can occur when a Stokes layer interacts with a curved surface. However, in general it is known that when such an interaction takes place a secondary steady streaming is set up. The nature of the steady streaming depends crucially on a steady streaming Reynolds number R_s whose importance was explained clearly by Stuart (1966). For large values of R_s the streaming decays to zero in a steady outer boundary layer of relative thickness $R_s^{1/2}$ whilst for small values of R_s the steady motion is confined to the Stokes layer. We shall see below that the instability mechanism described in Section 2 occurs for $R_s \gg 1$ and has an $O(1)$ effect on the steady streaming boundary layer.

The possibility that the instability described in Section 2 could occur in spatially varying oscillatory flows over curved walls was overlooked until Honji (1981) investigated the classical steady streaming flow induced by oscillating a cylinder transversely along its axis. In addition Honji investigated flows over wavy walls and over steps and found that a vortex instability of these time-periodic flows occurred at sufficiently large amplitudes of oscillation of the fluid velocity at infinity. We will discuss only the transversely oscillating cylinder problem in detail; related results for wavy walls, pipe flows, and spheres will be mentioned briefly later.

In his experiment Honji (1981) investigated the classical steady streaming problem of Schlichting (1932), Stuart (1966), and Riley (1967). For sufficiently small amplitude high frequency oscillations, he found that the flow remained two-dimensional and was consistent with theoretical predictions. However, at a critical frequency dependent amplitude of

oscillation the flow became three-dimensional in the neighbourhood of the positions on the cylinder where the tangent plane was parallel to the direction of motion of the cylinder. At a still higher critical amplitude of oscillation, Honji found that the flow became 'turbulent and separated.' We will show below that a linear stability analysis predicts the first critical amplitude of oscillation whilst a nonlinear theory suggests that a finite amplitude instability might cause the basic steady streaming boundary layer to separate prematurely. The details of the calculation outlined below can be found in the paper by Hall (1984).

Suppose that a circular cylinder of radius a oscillates with velocity $U_0 \cos \omega t$ along a diameter in a fluid of viscosity ν. The parameters which govern the two-dimensional flow are

$$\beta = \frac{\omega a^2}{\nu}, \tag{3.1a}$$

$$\lambda = \frac{U_0}{\omega a}, \tag{3.1b}$$

$$R_s = \frac{U_0^2}{\omega \nu} = \lambda^2 \beta. \tag{3.1c}$$

The frequency parameter β is taken to be large so that the boundary layer on the cylinder is thin compared to its radius whilst the amplitude parameter λ is taken to be small. Before specifying the size of the steady streaming Reynolds number R_s we write down the Taylor number T based on $(\nu/\omega)^{1/2}$ the boundary layer thickness. We obtain

$$T = \frac{2^{3/2} U_0^2}{a\nu^{1/2}\omega^{3/2}},$$

$$= 2^{3/2} R_s \beta^{-1/2},$$

so that if instability occurs at $O(1)$ values of T we must take R_s formally to be $O(\beta^{-1/2})$; thus we take the limit $\beta \to \infty$ with $R_s \beta^{-1/2}$ held fixed. We further note that in this limit $\lambda \sim \beta^{-1/4}$ so that the boundary layer on the cylinder is essentially a Stokes layer. In Figure 2 we have sketched the cylinder and the various regions of interest. The inviscid slip velocity for the basic flow has maxima at $\theta = \pm \pi/2$, and since the curvature of the boundary is constant we expect that these will be the least stable locations. A WKB approach to the instability problem quickly shows that $\theta = \pm \pi/2$ are turning points of the expansion. In fact, they are second-order turning points so that an inner region of angle $O(\beta^{-1/8})$ is needed at $\theta = \pm \pi/2$.

Suppose then that the basic flow is perturbed to a disturbance periodic along the axis of the cylinder with wavelength $\frac{2\pi}{k}$. The linear stability equations can be reduced to

$$L\left(\frac{\partial^2}{\partial\eta^2} - k^2\right)U = 2k^2 T \sin\theta\, \bar{v}_0 V - \frac{2^{5/4} \sin\theta\, \bar{v}_{0\eta\eta} T^{1/2}}{\beta^{1/4}} \frac{\partial U}{\partial\theta} + O(\beta^{-1/4}),$$

$$LV = 4 \sin\theta \frac{\partial \bar{v}_0}{\partial\eta} U + O(\beta^{-1/4}), \tag{3.2}$$

where

$$L \equiv \frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau} - \frac{2^{5/4} T^{1/2} \sin\theta\, \bar{v}_0}{\beta^{1/4}} \frac{\partial}{\partial\theta}$$

with

$$\bar{v}_0 = \cos\tau - \cos(\tau - \eta)e^{-\eta}. \tag{3.3}$$

The Taylor number T is expanded as

$$T = T_0 + \beta^{-1/4} T_1 + \cdots$$

whilst near $\theta = \pi/2$ the velocity components U, and V expand as

$$U = U_0(\eta, \tau, \Phi) + \beta^{-1/8} U_1(\eta, \tau, \Phi) + \cdots$$

$$V = V_0(\eta, \tau, \Phi) + \beta^{-1/8} V_1(\eta, \tau, \Phi) + \cdots$$

where $\Phi = \beta^{1/8}(\theta - \pi/2)$. The functions U_0, V_0 can be written

$$(U_0, V_0) = A(\Phi)(u_0(\eta, \tau), v_0(\eta, \tau))$$

where A is an amplitude function to be determined whilst (u_0, v_0) satisfy

$$\begin{aligned}
&\left\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - k^2\right\}u_0 = 2k^2 T_0 \bar{v}_0 v_0, \\
&\left\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right\}v_0 = 4\frac{\partial\bar{v}_0}{\partial\eta} u_0, \\
&u_0 = \frac{\partial u_0}{\partial\eta} = v_0 = 0, \quad \eta = 0, \\
&u_0, v_0 \to 0, \ \eta \to \infty.
\end{aligned} \tag{3.4}$$

This eigenvalue problem was solved by Hall (1984) in exactly the same manner as that described in Section 2. The neutral curve $k = k(T_0)$ is shown in Figure 3; we see that instability is predicted locally near $\theta = \pi/2$ for

$$T_0 > 11.99. \tag{3.5}$$

The amplitude function $A(\Phi)$ is found at higher order to satisfy

$$\frac{d^2 A}{d\Phi^2} + \mu[T_1 - \Phi^2]A = 0. \tag{3.6}$$

Here μ is a positive constant which must be calculated numerically; the solutions of this equation, which decay when $\Phi \to \pm \pi/2$ are

$$A = U_n(-n - \tfrac{1}{2}\,,\ 2\mu^{1/4}\,\Phi)$$

where U_n is the nth parabolic cylinder function and

$$T_1 = T_{1n} = \frac{2[n + \tfrac{1}{2}]}{\sqrt{\mu}}\,.$$

The most unstable mode corresponds to $n = 0$ and gives

$$A_0 = e^{-\mu^{1/2}\,\Phi^2/2},$$

with

$$T_{10} = \frac{1}{\sqrt{\mu}} = 5.51.$$

Thus the critical value of R_s is

$$R_s = R_{sc} = 4.24[\beta^{1/2} + .46\beta^{1/4} + \cdots], \tag{3.7}$$

or in terms of λ the critical oscillation amplitude corresponds to

$$\lambda = \lambda_c = \frac{2.06}{\beta^{1/4}}\,[1 + \frac{.23}{\beta^{1/4}} + \cdots]. \tag{3.8}$$

In Figure 4 we have plotted the results predicted by (3.8) and Honji's results; there seems little doubt that the instability mechanism is responsible for the onset of the three-dimensionality described by Honji.

The development of this instability into a weakly nonlinear region leads to some surprising results. We choose to write the Taylor number T in the form

$$T = T_{0c} + \beta^{-1/4}\overline{T}$$

where T_{0c} is the critical value of T_0. A Stuart-Watson expansion with the fundamental terms of order $\beta^{-1/8}$ shows that the generalization of (3.6) into the nonlinear stage is

$$\frac{d^2 A}{d\Phi^2} + \mu[\overline{T} - \Phi^2]A = \gamma A^3. \tag{3.9}$$

Numerical calculations by Hall (1984) showed that γ is negative so that finite amplitude solutions bifurcate subcritically at the critical Taylor number. The determination of γ requires the solution of several partial differential systems so the possibility that a numerical error was made should not be overlooked. However, if γ is indeed negative then it suggests that experimentally three-dimensionality could be induced by sufficiently large

subcritical perturbations. The nonlinear theory shows that the steady streaming boundary layer is, at least in an $0(\beta^{-1/8})$ neighbourhood of $\theta = \pi/2$, driven by both the instability and the basic oscillatory flow. If we define an outer boundary layer variable ξ by

$$\xi = (\frac{r - a}{a})\beta^{1/4}$$

then the outer steady boundary layer is given by

$$v = \frac{\nu}{a}\beta^{3/8}\psi_\xi, \quad u = \frac{-\nu}{a}\beta^{1/4}\psi_\Phi$$

with

$$\psi_{\xi\xi\xi\xi} = \psi_\xi \psi_{\xi\xi\Phi} - \psi_\Phi \psi_{\xi\xi\xi}, \tag{3.10}$$

which must be solved subject to

$$\begin{aligned} &\psi = 0, \quad \xi = 0 \\ &\psi_\xi \to 0, \quad \xi \to \infty \\ &\psi_\xi \to T_{0c}\left[\frac{3\Phi}{2^{3/2}} - 4.39 \cdot 2^{3/4}\, T_{0c}^{-1/2} A \frac{dA}{d\Phi}\right], \quad \xi \to 0. \end{aligned} \tag{3.11}$$

If we set $A = 0$ above we obtain the equations governing the attachment of a steady streaming boundary layer within a $\beta^{-1/8}$ neighbourhood of $\theta = \pi/2$. In that case (3.10) is solved subject to

$$\psi_\xi = 0, \quad \Phi = 0, \tag{3.12}$$

and the symmetry of (3.9) about $\Phi = 0$ means that (3.12) can still be applied since either A or $\frac{dA}{d\Phi} = 0$ at $\Phi = 0$. We note that for large Φ the condition (3.11) reduces to $\psi_\xi \to T_{0c} \frac{3\Phi}{2^{3/2}}$, $\xi \to 0$, so that, assuming that the boundary layer remains attached for finite values of Φ, the extra term proportional to $A \frac{dA}{d\Phi}$ in (3.11) merely produces an origin shift in the large Φ asymptotic solution of (3.10). The linear eigenfunctions $A_n(\Phi)$ for $n \neq 0$ all have intervals where $A_n(\Phi)A_n^1(\Phi)$ is positive so that the possibility exists that in the nonlinear regime the slip velocity in (3.11) might change sign at some value of Φ. If the magnitude of the inviscid slip velocity is sufficiently large where this occurs, then the attached flow strategy fails and the steady streaming boundary layer will detach prematurely from the cylinder. This possibility does not occur for more general flows where the point of attachment of the steady streaming layer and the most unstable position do not coincide. In this case the steady streaming driven by the instability is weak compared to that of the basic flow.

For more general steady streaming flows Papageorgiu (1985) has shown that both concave and convex curvature lead to this local vortex type of instability. Consider then an oscillatory viscous flow adjacent to the wall $y = 0$ induced by a outer potential flow with slip velocity $U_0\, U(x)\cos \omega t$. If the wall has radius of curvature $aR(x)$, we write down the local Taylor number T_ℓ defined by

$$T_\ell = \frac{2^{3/2}\, U_0^2\, U^2(x)}{a\nu^{1/2}\, \omega^{3/2}\, R(x)} \cdot$$

The mechanism described above for the circular cylinder problem occurs near $x = x_m$ if

$$T_\ell(x_m) \sim 11.99,$$

$$\frac{d}{dx} T_\ell = 0, \quad x = x_m.$$

Hence with variable curvature the most unstable locations do not necessarily occur where the inviscid slip velocity has a maximum; this produces some interesting results. Thus for an ellipse oscillating transversly there can be two or six local regions of instability depending on the angle of attack and the eccentricity of the ellipse.

Papageorgiu has shown that for concave curvature the corresponding condition is

$$|T_\ell| = 7.104,$$

$$\frac{d}{dx} |T_\ell| = 0, \quad x = x_m$$

so that a locally concave wall is more susceptible to instability than is a convex one. The result that either a concave or a convex wall lead to instability is quite different than the classical results for steady flows; however, for a time-dependent flow Rayleigh's criterion does not apply so there is no reason why this should not be the case.

The implications of Papageorgiu's calculations for flows in curved pipes and over wavy walls are important. For a curved pipe he has shown that an oscillating azimuthal pressure gradient leads to instability at both the inner and outer bends; however, the outer bend is the most unstable location. This instability leads to rapid local variations in the shear stress at the wall which might have important consequences for aortic blood flow. It should also

be pointed out that in this problem the turning point structure changes from that found for the circular cylinder problem; the technical problems associated with this change were overcome using the expansion procedure devised by Soward and Jones (1981). A similar difficulty arises when the instability of the flow due to a torsionally oscillating sphere is considered; in this problem the instability is localized near the equation. As yet the generalization of the expansion procedure of Soward and Jones into the nonlinear regime has not been given.

4. THE LINEAR STABILITY EQUATIONS FOR THE FLOW AROUND A RAPIDLY ROTATING CYLINDER

Consider the flow of a viscous fluid of kinematic viscosity ν around a circular cylinder of radius a. The cylinder rotates with angular velocity $\frac{V}{a}$ whilst the flow a long way from the cylinder has sped U in the x direction. We follow the notation of Moore (1957) and define the parameter ε by

$$\varepsilon = \frac{U}{V} , \tag{4.1}$$

and a Reynolds number R by

$$R = \frac{Va}{\nu} . \tag{4.2}$$

Moore has discussed the above flow in the limit $\varepsilon \to 0$; here we will also assume that $R \gg 1$. For large values of R it is known from the work of Glauert (1957), Moore (1957), and Wood (1957) that a boundary layer of thickness $aR^{-1/2}$ is set up on the surface of the cylinder. Here we will

consider the possible instability of this boundary layer to a Taylor-Görtler vortex type of perturbation. An examination of the effective Taylor-Görtler number for such a flow suggests that instability is likely for $O(1)$ values of the parameter

$$T = \varepsilon R^{1/2}; \tag{4.3}$$

hence we limit our investigation to the limit $\varepsilon \to 0$ with T, which we will refer to as the Taylor number, held fixed. On the assumption that $\varepsilon \sim R^{-1/2}$ we can show from the work of Moore (1957) that in the boundary layer u and v, the radial and azimuthal velocity components, are given by

$$u = V\varepsilon R^{-1/2}\{\bar{u}(\eta,\ \theta) + O(R^{-1/2})\}, \tag{4.4a}$$

$$v = V\{1 + \varepsilon \bar{v}(\eta,\ \theta) + O(R^{-1})\}. \tag{4.4b}$$

Here η is a boundary layer variable defined by

$$\eta = \frac{(r - a)}{a}\left(\frac{R}{2}\right)^{1/2} \tag{4.5}$$

whilst $\bar{u}$ and $\bar{v}$ can be obtained from Moore's work; in fact, we shall only need an explicit form for $\bar{v}$ which is given by

$$\bar{v} = ie^{i\theta-\eta(1+i)} + \text{complex conjugate}. \tag{4.6}$$

Thus $\bar{v}$ is just the Stokes layer velocity profile with θ taking the place of the time variable. It is this correction to the irrotational flow $v = 1/r$

which leads to instability. This result follows from the fact that the irrotational flow $v \sim 1/r$ is neutrally stable according to Rayleigh's criterion.

Suppose that we define the dimensionless variables τ and Z by

$$Z = \sqrt{\frac{R}{2}}\,\frac{z}{a}\,, \quad \tau = \frac{Vt}{a}\,,$$

where z represents distance along the axis of the cylinder and t denotes time. The flow given by (4.4) is perturbed by writing

$$\frac{u}{V} = R^{-1/2}[\bar{u} + O(R^{-1/2})] + U(\eta,\ \theta,\ \tau)e^{ikZ}, \tag{4.7a}$$

$$\frac{v}{V} = [1 + \bar{v}(\eta,\ \theta) - \sqrt{\frac{2}{R}}\,\eta + O(R^{-1})] + V(\eta,\ \theta,\ \tau)e^{ikZ}, \tag{4.7b}$$

$$\frac{w}{V} = W(\eta,\ \theta,\ z)e^{ikZ}. \tag{4.7c}$$

Thus k is the wavenumber of the vortex type of instability and, unlike the usual case with a centrifugal instability mechanism, the three components of the distrubance velocity field are of comparable size. The disturbed velocity field (4.7) can be substituted into the Navier-Stokes equations, and after some manipulation we obtain the linear stability equations:

$$\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\,\frac{\partial}{\partial\theta} - 2\,\frac{\partial}{\partial\tau}\}U = \sqrt{2}\,\frac{\partial P}{\partial\eta} - 4v, \tag{4.8a}$$

$$\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\,\frac{\partial}{\partial\theta} - 2\,\frac{\partial}{\partial\tau}\}V = \sqrt{2}\ TU\,\frac{\partial\bar{v}}{\partial\eta}, \tag{4.8b}$$

$$\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\theta} - 2\frac{\partial}{\partial\tau}\}W = ikP\sqrt{2}. \tag{4.8c}$$

Here P is the pressure perturbation scaled on $\rho\nu R^{-1/2}v^2$ and terms of relative order $R^{-1/2}$ have been neglected. The continuity equation corresponding to (4.8) is

$$\frac{\partial U}{\partial\eta} + ikW = 0. \tag{4.9}$$

It is convenient at this stage to eliminate P and W from (4.8) and (4.9) to give the coupled pair of equations:

$$\begin{aligned}
&\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\ell} - 2\frac{\partial}{\partial\tau}\}\{\frac{\partial^2}{\partial\eta^2} - k^2\}U = 4k^2\, V,\\
&\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\ell} - 2\frac{\partial}{\partial\tau}\}V = \sqrt{2}\; TU\,\frac{\partial\bar{v}}{\partial\eta},
\end{aligned} \tag{4.10}$$

which must be solved subject to

$$V = U = \frac{\partial U}{\partial\eta} = 0, \quad \eta = 0, \infty.$$

Thus we assume that the instability is confined to the boundary layer; this is to be expected so long as we ignore nonlinear effects.

Suppose next that a viscous fluid of Prandtl number unity and viscosity ν occupies the region $z \geq 0$ and that the plane $z = 0$ has temperature $2T_0 \cos \omega t$. We define dimensionless variables τ and η by

$$\tau = \omega t, \tag{4.11a}$$

$$\eta = \left\{\frac{\omega}{2\nu}\right\}^{1/2} z \tag{4.11b}$$

so that the motionless state has temperature field T given by

$$T = T_0\, \bar{v}(\tau, \eta) \tag{4.12}$$

where $\bar{v}$ is as defined by (4.6) with θ replaced by τ. Thus the temperature is zero except for a thin region of thickness $\Delta = \left\{\frac{\omega}{2\nu}\right\}^{-1/2}$ whilst in the boundary layer the temperature is given by the Stokes layer velocity profile. In the usual way the basic state is now perturbed by a disturbance with horizontal wavenumber k scaled on Δ^{-1}. If the Boussinesq approximation is made then, following Chandrasehkar (1963), Chapter II, we obtain

$$\left\{\frac{\partial^2}{\partial \eta^2} - k^2\right\}\left\{\frac{\partial^2}{\partial \eta^2} - k^2 - 2\frac{\partial}{\partial \tau}\right\} W = R\Theta, \tag{4.13a}$$

$$\left\{\frac{\partial^2}{\partial \eta^2} - k^2 - 2\frac{\partial}{\partial \tau}\right\}\Theta = \frac{\partial \bar{v}}{\partial \eta} W, \tag{4.13b}$$

where Θ is the temperature perturbation and W the z velocity component of the perturbation. The parameter R appearing in (4.13) is the Rayleigh number defined by

$$R = g\alpha\Delta^3\, T_0\, \nu^{-2} \tag{4.14}$$

where α is the coefficient of volume expansion. If the plane $\eta = 0$ is rigid then (4.13) are to be solved subject to

$$\Theta = W = \frac{\partial W}{\partial \eta} = 0, \quad \eta = 0, \infty. \tag{4.15}$$

Thus if we seek neutral solutions of (4.10) with $\frac{\partial}{\partial \tau} = 0$ we see that the resulting eigenvalue problem is identical to that given by (4.13), (4.15) if we associate T with $R/4\sqrt{2}$ and θ with τ respectively. Thus there exists an analogy between the neutral stability problems for the flow past a rapidly rotating cylinder and a time-periodically heated fluid layer. The analogy corresponds to that between the steady Taylor vortex instability between circular cylinders rotating at almost the same speed and Bénard convection between rigid walls. The eigenvalue problem (4.13)-(4.14) can be solved using the method outlined in Section 2; so far we have found one mode of instability with a minimum value of $R \sim 54$. However, other modes exist, but it is not yet clear which is the most unstable.

5. CONCLUSION

The instability mechanism found several years ago for the flow around a torsionally oscillating cylinder exists in many flows of practical importance. When the basic oscillatory flow which supports the instability is spatially varying, the instability becomes spatially concentrated near the most unstable positions. Moreover the instability drives the secondary steady streaming boundary layer at the same order as does the basic oscillatory flow. Recently it has been shown by Hall (1985) that this mechanism can lead to the instability of high frequency Tollmien-Schlichting waves interacting with a wall of either convex or concave curvature.

There are many flows where the basic state has a steady component of the same order as the oscillatory part, in such flows there exists the possibility of interactions between the instability mechanisms associated with the steady

and unsteady components. An obvious example of this is the Taylor vortex problem with a fixed outer cylinder and an inner cylinder rotating with angular velocity $\Omega[1 + \varepsilon\cos\omega t]$. For $\varepsilon \ll 1$, the flow is susceptible to steady Taylor vortices at sufficiently large vaules of Ω whilst for $\varepsilon \gg 1$, and $\omega \gg 1$ the Stokes layer mode is possible. For intermediate values of ε the modes interact in an as yet undetermined manner; the ratio of the axial lengthscales for the different modes is large so that some progress could probably be made asymptotically.

Another interaction problem yet to be investigated is that between the Tollmien-Schlichting and Taylor-Görtler modes of instability. Experimentally it is known that local transition in the flat Stokes layer takes place as Reynolds numbers as low as 300. It follows from (2.2) that for Stokes layers on curved walls with local radius of curvature R such that

$$\frac{\left(\frac{\nu}{\omega}\right)^{1/2}}{R} > \sim 10^{-3}$$

the centrifugal mode will occur first. This effectively means that in most situations the mode is likely to be dominant.

The preliminary results, which we have obtained so far for the sinsuiodally heated fluid layer, suggest that a convection mode of instability similar to that found by Seminara and Hall (1976) exists for that flow. As yet no experimental investigations of this problem have been made; it will be interesting to see how this flow evolves at high Rayleigh numbers.

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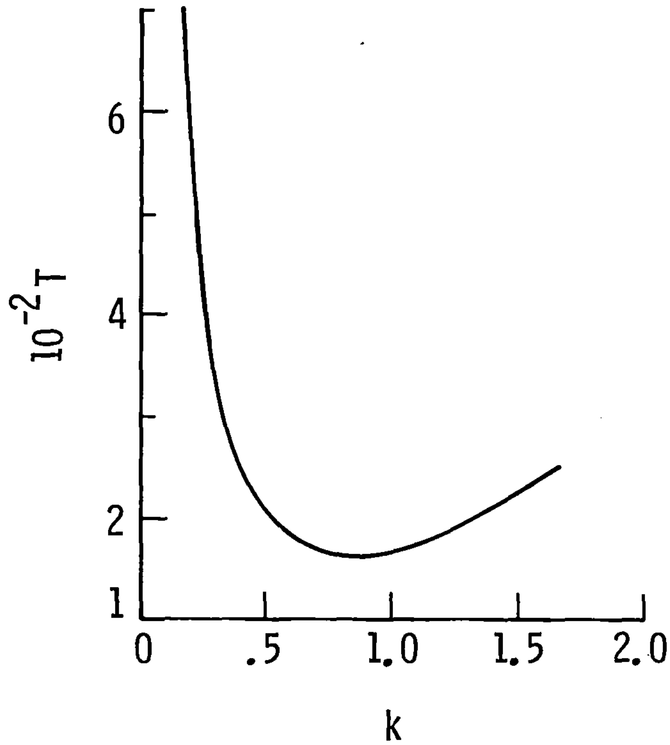

Figure 1

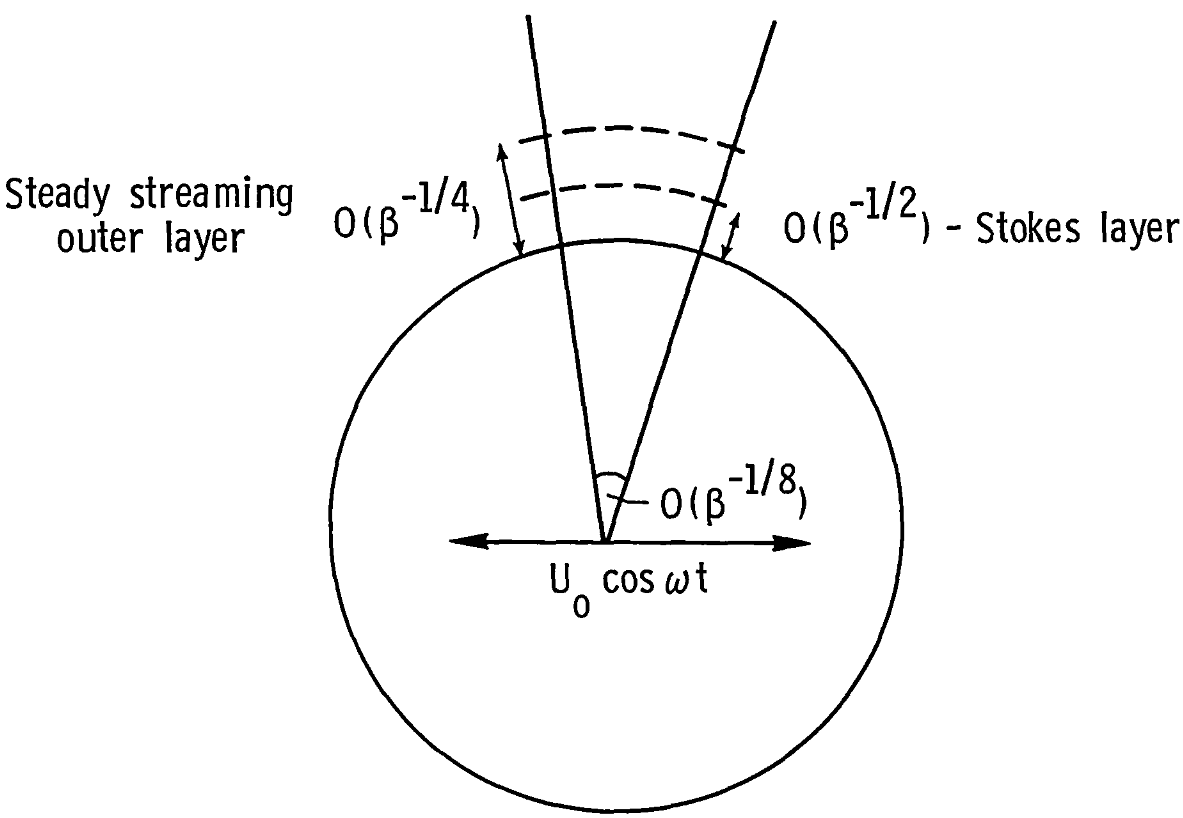

Figure 2

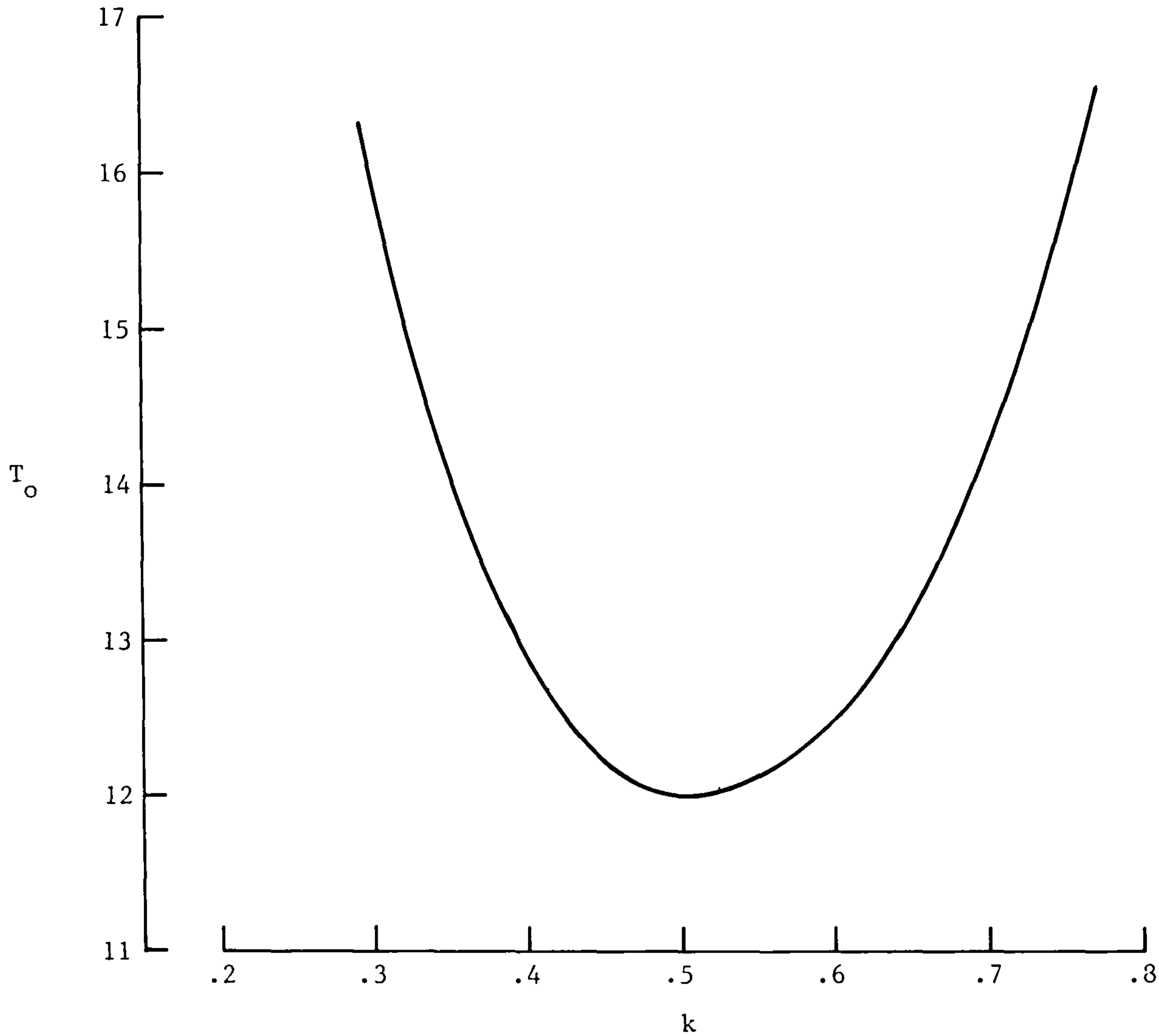

Figure 3

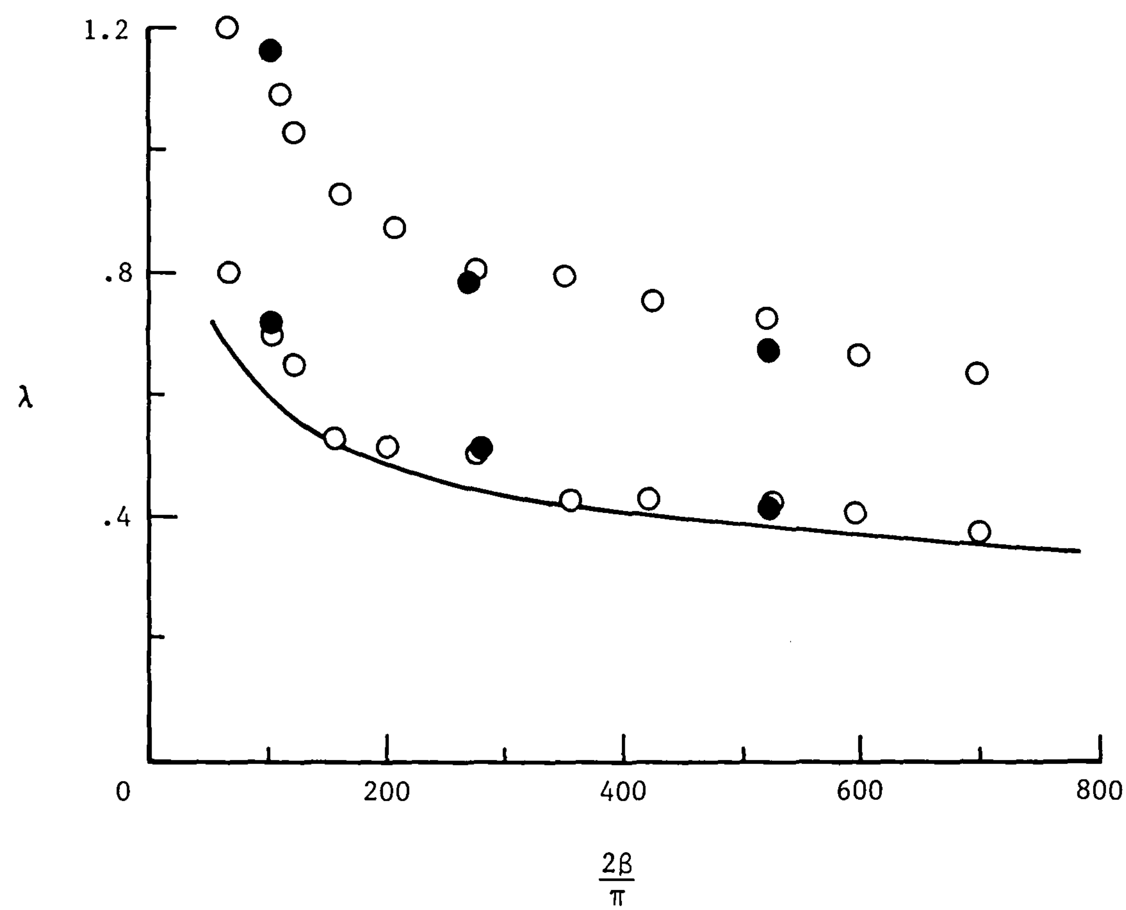

Figure 4

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15. Supplementary Notes

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Final Report

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16. Abstract

The instabilities of some spatially and/or time-periodic flows are discussed, in particular, flows with curved streamlines which can support Taylor-Görtler vortices are described in detail. The simplest flow where this type of instability can occur is that due to the torsional oscillations of an infinitely long circular cylinder. For more complicated spatially varying time-periodic flows, a similar type of instability can occur and is spatially localized near the most unstable positions. When nonlinear effects are considered it is found that the instability modifies the steady streaming boundary layer induced by the oscillatory motion. It is shown that a rapidly rotating cylinder in a uniform flow is susceptible to a related type of instability; the appropriate stability equations are shown to be identical to those which govern the instability of a Boussinesq fluid of Prandtl number unity heated time periodically from below.

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